Math 247A Lecture 17 Notes

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1 L^p Bounds for Calderón-Zygmund Convolution Kernels

1.1 Weak L^p bound for Calderón-Zygmund convolution kernels

Theorem 1.1. *Let $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ be a Calderón-Zygmund convolution kernel. For $\varepsilon > 0$, let $K_\varepsilon = K \mathbb{1}_{\{\varepsilon \leq |x| \leq 1/\varepsilon\}}$. Then*

1. *$|\{x : |K_\varepsilon * f|(x) > \lambda\}| \lesssim \frac{1}{\lambda}\|f\|_1$ uniformly in $\lambda > 0, f \in L^1, \varepsilon > 0$.*
2. *For any $1 < p < \infty$, $\|K_\varepsilon * f\|_p \lesssim \|f\|_p$ uniformly for $f \in L^p, \varepsilon > 0$.*

Consequently, $f \mapsto K * f$ (the L^p-limit of $K_\varepsilon * f$) extends continuoussly from $\mathcal{S}(\mathbb{R}^d)$ to a bounded map on L^p when $1 < p < \infty$.

Proof. Assuming that (1) holds, we proved (2) using interpolation and duality. To show the last claim, it suffices to prove that $\{K_\varepsilon * f\}_{\varepsilon > 0}$ forms a Cauchy sequence in L^p $(1 < p < \infty)$ whenever $f \in \mathcal{S}(\mathbb{R}^d)$. We want to prove this using the L^2 result and condition (c) of the Calderón-Zygmund kernel; this will let our theory have more adaptability.

For $1 < p < 2$, let $1 < q < p$. Write $\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{2}$ for some $\theta \in (0, 1)$. Then

$$\|K_{\varepsilon_1} * f - K_{\varepsilon_2} * f\|_p \lesssim \underbrace{\|K_{\varepsilon_1} * f + K_{\varepsilon_2} f\|_2^{1-\theta}}_{\xrightarrow{\varepsilon_1, \varepsilon_2 \to 0} 0} \underbrace{\|K_{\varepsilon_1} * f + K_{\varepsilon_2} f\|_q^{\theta}}_{\leq (\|K_{\varepsilon_1} * f\|_q + \|K_{\varepsilon_2} * f\|_q)^\theta \lesssim \|f\|_q^\theta}$$
$$\xrightarrow{\varepsilon_1, \varepsilon_2 \to 0} 0.$$

For $2 < p < \infty$¡ let $p < r < \infty$ and write $\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{r}$. Then

$$\|K_{\varepsilon_1} * f - K_{\varepsilon_2} * f\|_p \leq \underbrace{\|K_{\varepsilon_1} * f - K_{\varepsilon_2} * f\|_2^{1-\theta}}_{\xrightarrow{\varepsilon_1, \varepsilon_2 \to 0} 0} \underbrace{\|K_{\varepsilon_1} * f - K_{\varepsilon_2} * f\|_r^{\theta}}_{\lesssim \|f\|_r^\theta}$$

Let's show (1). For $\lambda > 0$, $f \in L^1$, and $\varepsilon > 0$, perform a Calderón-Zygmund decomposition for f at level λ: $f = g + b$ with $\operatorname{supp} b = \bigcup Q_k$, Q_k^o pairwise disjoint, and

$\sum_k |Q_k| \le \|f\|_1/\lambda$. We can take

$$g(x) = \begin{cases} f(x) & x \notin \bigcup Q_k \\ \frac{1}{|Q_k|}\int_{Q_k} f(y)\,dy & x \in Q_k^o. \end{cases}$$

Then $|g| \lesssim \lambda$, and $b(x) = f(x) - \frac{1}{|Q_k|}\int_{Q_k} f(y)\,dy$ for $x \in Q_k$, so

$$\int_{Q_k} b(x)\,dx = 0, \qquad \frac{1}{|Q_k|}\int_{Q_k} |b(y)| \lesssim \lambda.$$

Then

$$\begin{aligned} |\{x : |K_\varepsilon * f|(x) > \lambda\}| &\le |\{x : |K_\varepsilon * g|(x) > \lambda/2\}| + |\{x : |K_\varepsilon * b|(x) > \lambda/2\}| \\ &\lesssim \frac{1}{\lambda^2}\|K_\varepsilon * g\|_2^2 + \left|\bigcup_k \alpha Q_k\right| + \left|\{x \in \left[\bigcup \alpha Q_k\right]^c : |K_\varepsilon * b|(x) > \lambda/2\}\right| \end{aligned}$$

We have

$$\frac{1}{\lambda^2}\|K_\varepsilon * g\|_2^2 \lesssim \frac{\|g\|_2^2}{\lambda^2} \lesssim \frac{\lambda\|g\|_1}{\lambda^2} \lesssim \frac{\|f\|_1}{\lambda}$$

and

$$\left|\bigcup \alpha Q_k\right| \le \sum |\alpha Q_k| \le \alpha^d \sum |Q_k| \lesssim \alpha^d \frac{\|f\|_1}{\lambda}.$$

We are left with $E := |\{x \in [\bigcup \alpha Q_k]^c : |K_\varepsilon * b|(x) > \lambda/2\}|$. Let $x \notin \bigcup \alpha Q_k$. Then

$$\begin{aligned} K_\varepsilon * b(x) &= \int K_\varepsilon(x-y)b(y)\,dy \\ &= \sum_k \int_{Q_k} K_\varepsilon(x-y)b(y)\,dy \end{aligned}$$

Here, we only have a convolution, not an average. But a convolution is only as smooth as its smoothest term. So we have to use the regularity of K_ε (condition (c)).

$$= \sum_k \int_{Q_k} [K_\varepsilon(x-y) - K_\varepsilon(x-x_k)]b(y)\,dy.$$

Using Chebyshev,

$$\begin{aligned} E &\lesssim \frac{1}{\lambda}\int_{x\notin\bigcup \alpha Q_k} (K_\varepsilon * b)(x) \\ &\lesssim \frac{1}{\lambda}\sum_k \int_{x\in(\alpha Q_k)^c}\int_{Q_k} |K_\varepsilon(x-y) - K_\varepsilon(x-x_k)||b(y)|\,dy\,dx \end{aligned}$$

Change variables.

$$\lesssim \frac{1}{\lambda}\sum_k \int_{Q_k} |b(y)| \left(\int_{(\alpha Q_k)^c - \{x_k\}} |K_\varepsilon(x + x_k - y) - K_\varepsilon(x)|\, dx \right) dy.$$

For $y \in Q_k$, $|x_k - y| \le \frac{1}{2}\ell(Q_k)\sqrt{d}$. So we need $\alpha\ell(Q_k)/2 \ge 2\frac{1}{2}\ell(Q_k)\sqrt{d}$. So take $\alpha \ge 2\sqrt{d}$. Then using the regularity condition (c) of the convolution kernel, we get

$$\begin{aligned} E &\lesssim \frac{1}{\lambda}\sum_k \int_{Q_k} |b(y)| \cdot 1\, dy \\ &\lesssim \frac{\|f\|_1}{\lambda}. \end{aligned}$$

□

Remark 1.1. Once we have boundedness in L^2, the only condition we need to deduce boundedness in L^p for $1 < p < \infty$ is the regularity condition (c).

1.2 Application: The Hilbert transform

Here is an application.

Example 1.1. Let $K : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be $K(x) = \frac{1}{\pi x}$. This is a Calderón-Zygmund convolution kernel. So the **Hilbert transform**,

$$Hf(x) = \frac{1}{\pi}\int \frac{f(x-y)}{y}\, dy = \lim_{\varepsilon \to 0} \int_{|y| > \varepsilon} \frac{f(x-y)}{y}\, dy.,$$

is bounded on L^p for $1 < p < \infty$.

Remark 1.2. Boundedness on L^1 and L^∞ may fail. Consider the Hilbert transofrm, and take $f = \mathbb{1}_{[a,b]} \in L^1 \cap L^\infty$; we will show that $Hf \notin L^1 \cup L^\infty$. For $\varepsilon > 0$,

$$\begin{aligned} H_\varepsilon f(x) &:= \frac{1}{\pi}\int_{\varepsilon \le |y| \le 1/\varepsilon} \frac{\mathbb{1}_{[a,b]}(x-y)}{y}\, dy \\ &= \frac{1}{\pi}\int_{\substack{\varepsilon \le |y| \le 1/\varepsilon \\ x-b \le y \le x-a}} \frac{1}{y}\, dy \\ &= \frac{1}{\pi}\log\left|\frac{x-a}{x-b}\right| \end{aligned}$$

almost everywhere. But $Hf \notin L^1 \cup L^\infty$.

Remark 1.3. We have $\widehat{Hf}(\xi) = -i\,\mathrm{sgn}(\xi)\cdot \widehat{f}(\xi)$. For $a > 0$, let

$$\widehat{f}_a(\xi) = \begin{cases} e^{-a\xi} & \xi \ge 0 \\ 0 & \xi < 0, \end{cases} \qquad \widehat{g}_a(\xi) = \begin{cases} 0 & \xi > 0 \\ e^{q\xi} & \xi \le 0. \end{cases}$$

Then

$$(\widehat{f_a} - \widehat{g}_a)(\xi) = \begin{cases} e^{-a\xi} & \xi > 0 \\ 0 & \xi = 0 \\ -e^{a\xi} & \xi < 0 \end{cases} \xrightarrow{\mathcal{S}'(\mathbb{R}),\, a \to 0} \begin{cases} 1 & \xi > 0 \\ 0 & \xi = 0 \\ -1 & \xi < 0 \end{cases} = \operatorname{sgn}(\xi).$$

So we get

$$f_a - g_a \xrightarrow{\mathcal{S}'(\mathbb{R}),\, a \to 0} \operatorname{sgn}^\vee.$$

Next time, we will complete this computation.